On the Einstein-Podolsky-Rosen Proof of the 'Incompleteness' of Quantum Mechanics

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Abstract

It is shown that the Einstein-Podolsky-Rosen conclusion concerning the 'incompleteness' of Quantum Mechanics does not follow from the results of their proposed gedanken experiment, but is rather stated as a premise. If it were possible to perform the experiment it would, in fact, show that Quantum Mechanics is 'complete' for the observables discussed. Because, however, of the non square-integrable nature of the wave function, the proposed experiment gives vanishing probabilities for measurements performed in finite intervals of configuration or momentum space. Hence no conclusion as to the 'completeness', or otherwise, of Quantum Mechanics can be drawn from the experiment.

Perhaps no other paper written in the present century has generated as much debate about questions related to the foundations of physics and their philosophical implications than that of Einstein, Podolsky and Rosen (EPR) [1]. However, after the initial replies written by Bohr [2], Furry [3] and Schrödinger [4], there has been very little critical discussion of the EPR paper itself in the literature [5]. In this letter a reappraisal of the EPR paper is made and the following conclusions are drawn:

(i) The argument presented by EPR to demonstrate the 'incompleteness' of Quantum Mechanics (QM) is invalidated by a logical error.

(ii) The gedanken experiment proposed by EPR cannot be carried out if the usual probabilistic interpretation of QM is correct, and so no physical conclusions can be drawn from the experiment.

Following EPR, a theory is said to give a 'complete' description of a physical quantity if the following condition is satisfied:

'Without, in any way, disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of the physical quantity'.

If this is the case, EPR associate an 'Element of Physical Reality' to the the corresponding quantity. EPR also require that, in a 'complete' theory:

'Every Element of Physical Reality must have a counterpart on the physical theory'.

This hypothesis is not particularly important since it must necessarily be true if the theory is able to predict the value of the corresponding physical quantity.

The EPR gedanken experiment will first be discussed from a purely logical viewpoint. Secondly, the conceptual feasiblity within QM, of the proposed experiment is examined. The following hypotheses are defined:

- QMT : QM is a true theory within its domain of applicablity.
- QMTC(A,B,..) : QM is a true, complete, theory for the physical quantities A,B,.. .
- PRNC(A,B) : Elements of Physical Reality exist for each of a pair of physical quantities A, B with non-commuting operators in QM.

The EPR gedanken experiment is based solely on the hypothesis QMT (Quantum Mechanics True). Contrary to the statement of EPR, it is <u>not necessary</u> to assume at the outset that QM is also a complete theory (hypothesis QMTC). In fact, applying QMT and assuming also that a quantum mechanical system of two correlated particles with a certain well-defined wave function can be constructed, EPR found that Elements of Physical Reality apparently <u>can</u> be assigned to each of the quantities P and Q that that have non-commuting operators. Using the symbol $\Rightarrow$ for 'logically implies' EPR then found that:

$$QMT \ \Rightarrow PRNC(P,Q)$$

After correction [6], the final statement of the result of the gedanken experiment is:

'Starting from the assumption of the correctness of QM (i.e. hypothesis QMT) we arrived at the conclusion that two physical quantities with non-commuting observables can have simultaneous reality.'

According to EPR's definitions, if two physical quantities have corresponding elements of physical reality, then the theory is a complete one for these quantities. In symbols [7]:

$$PRNC(P,Q) \otimes QMTC(P,Q) = TRUE \tag{1}$$

Using De Morgan's Theorem, (1) implies:

$$\overline{PRNC(P,Q)} \oplus \overline{QMTC(P,Q)} = FALSE \tag{2}$$

However, EPR state that the right side of (2) is TRUE and conclude, instead of (1), that:

$$PRNC(P,Q) \otimes QMTC(P,Q) = FALSE \tag{3}$$

Since the gedanken experiment showed that:

$$PRNC(P,Q) = TRUE,$$

EPR drew, on the basis of (3), the erroneous conclusion that:

$$QMTC(P,Q) = FALSE.$$

i.e. that QM is an incomplete theory. The basic assertion of EPR (actually, as shown above, in contradiction to the result of their gedanken experiment) is:

$$\overline{PRNC(P,Q)} \oplus \overline{QMTC(P,Q)} = TRUE \tag{4}$$

How is this assertion justified in the EPR paper? After discussion of quantum mechanical measurements on a *single particle*, with no obvious relevance to the case of *two correlated particles* as used in their gedanken experiment, EPR state that:

'From this it follows [1] the quantum mechanical description of reality given by the wave function is not complete or [2] when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality. For if both of them had simultaneous reality - and thus definite values - these values would enter into the complete description according to the condition of completeness. If the wave function provided such a complete description of reality it would contain these values, these would be predictable. This not being the case we are left with the alternatives stated.'

This argument, expressed symbolically by Eqn.(4), seems to be justified by the preceding discussion in the paper of non commuting observables (position and momentum) for a *single* particle, not the correlated two particle system of the gedanken experiment subsequently presented. In fact no justification is given by EPR for the application, *a priori* of propositions [1] and [2] to the gedanken experiment. Even so, one can still ask what is the meaning of EPR's assertion in Eqn.(4)? As quoted above, EPR carefully explain that the proposition [2] implies that the quantum mechanical description of two commuting observables is not complete, i.e.

$$[2] \equiv \overline{PRNC(A,B)} \Rightarrow \overline{QMTC(A,B)}$$

But proposition [1] *is* $\overline{QMTC(A,B)}$, so the assertion of EPR is actually 'either quantum mechanics is not complete or quantum mechanics not complete', so that their conclusion that quantum mechanics is not complete is inevitable! The assertion only become logically coherent in the case that 'not complete' in proposition [1] is replaced by 'complete' equivalent to the always correct assertion 'either quantum mechanics is complete or quantum mechanics not complete'.

Correcting this logical error, it might seem that the EPR experiment establishes the 'completeness' of quantum mechanics for the two non-commuting quantities P and Q. For this, however, it is necessary that the suggested gedanken experiment can, at least in principle, be performed. It will now be shown that this is not the case, so that no conclusion can be drawn as the the 'completeness', or otherwise, of quantum mechanics, by the arguments presented by EPR.

The spatial wave function of the correlated two particle system discussed by EPR is:

$$\Psi(x_1,x_2) = \int_{-\infty}^{\infty} \exp\frac{2\pi i}{h}(x_1 - x_2 + x_0)p dp = h\delta(x_1 - x_2 + x_0) \tag{5}$$

The probability that the particle 1 will be observed in the interval $a < x_1 < b$, for any position of the particle 2, can be written as:

$$\begin{aligned} P(a < x_1 < b) &= Lim(L \to \infty)\frac{\int_a^b dx_1 \int_{-\infty}^{\infty} dx_2 |\Psi(x_1,x_2)|^2}{\int_{-L}^{L} dx_1 \int_{-\infty}^{\infty} dx_2 |\Psi(x_1,x_2)|^2} \\ &= Lim(L \to \infty)\frac{b-a}{2L} = 0 \end{aligned} \tag{6}$$

The particle 1 cannot therefore be observed in any finite interval of x_1, and so the Q measurement suggested in the EPR gedanken experiment cannot be carried out.

By making Fourier transforms with respect to x_1 and x_2 the momentum wavefunction corresponding to (5) is found to be:

$$\Psi(p_1,p_2) = \frac{h^2}{2\pi}\exp\frac{2\pi i p_1 x_0}{h}\delta(p_1 + p_2) \tag{7}$$

The probability to observe p_1 in the range $p_a < p_1 < p_b$ for any value of p_2 is:

$$\begin{aligned} P(p_a < x_1 < p_b) &= Lim(p \to \infty)\frac{\int_{p_a}^{p_b} dp_1 \int_{-\infty}^{\infty} dp_2 |\Psi(p_1,p_2)|^2}{\int_{-p}^{p} dp_1 \int_{-\infty}^{\infty} dp_2 |\Psi(p_1,p_2)|^2} \\ &= Lim(p \to \infty)\frac{p_b - p_a}{2p} = 0 \end{aligned} \tag{8}$$

The momentum of particle 1 cannot be measured in any finite interval so that the proposed $p_2 = P = -p_1$ measurement of the EPR gedanken experiment cannot be carried out. In fact, the correlated two particle wave function proposed by EPR is not square integrable either in configuration or momentum space and so has no probabilistic interpretation in QM. The single particle wavefunction discussed by EPR has the same shortcoming. Hence the 'relative probability' $P(a, b)$ of EPR's Equation (6) also vanishes. While the statement that 'all values of the coordinate are equally probable' is true, it is also true that the absolute probablity to observe the particle in any finite interval is zero.

The EPR two particle wavefunction is now modified to render it square integrable so that the results of the gedanken experiment may be interpreted according to the usual rules of QM. The suggested 'minimally modified' wavefunction is:

$$\tilde{\Psi}(x_1, x_2) = \frac{1}{(\sqrt{2\pi}\sigma_x)^{\frac{1}{2}}} \exp\left(\frac{x_0^2 - 2x_1^2 - 2x_2^2}{16\sigma_x^2}\right) \delta(x_1 - x_2 + x_0) \tag{9}$$

Like the EPR wavefunction (5) $\tilde{\Psi}$ vanishes unless $x_2 = x_1 + x_0$, but it is square integrable and normalised:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\tilde{\Psi}(x_1, x_2)|^2 dx_1 dx_2 = 1 \tag{10}$$

The EPR wavefunction (5) is recovered in the limit $\sigma_x \to \infty$. Performing a double Fourier transform on Eqn(9) yields the corresponding momentum wave function:

$$\tilde{\Psi}(p_1, p_2) = \frac{1}{\pi\sigma_p} \exp\left(-\frac{(p_1 + p_2)^2}{2\sigma_p^2}\right) \exp\left(\frac{2\pi i p_1 x_0}{h}\right) \tag{11}$$

where

$$\sigma_p = h/4\pi\sigma_x$$

. The wavefunction (11) is also square integrable and normalised:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\tilde{\Psi}(p_1, p_2)|^2 dp_1 dp_2 = 1 \tag{12}$$

and the EPR wavefunction (7) is recovered in the limit $\sigma_x \to \infty$, $\sigma_p \to 0$. Now, performing the EPR gedanken experiment, using instead the wavefunctions (9) and (11) it becomes clear that it is no longer possible to associate 'Elements of Physical Reality' to the position Q and the momentum P of the second particle by performing measurements on the first one. The probability $\delta P(x_1)$ that the spatial position of the first particle lies in the interval δx_1 around x_1[8] is:

$$\delta P(x_1) = \frac{1}{\sqrt{\pi}\sigma_x} \exp\left(-\frac{x_0^2}{8\sigma_x^2}\right) \exp\left(-\frac{(2x_1 + x_0)^2}{8\sigma_x^2}\right) \delta x_1 \tag{13}$$

Because of the δ-function in the wave function (9), this is also the probability that x_2 lies in the interval of width $\delta x_2 = \delta x_1$ around $x_2 = x_1 + x_0$. Measuring x_1 in the interval δx_1 then enables the certain prediction that x_2 lies in the interval δx_2 around $x_2 = x_1 + x_0$. However, to associate an 'Element of Physical Reality' to x_2 requires that the *value* must be exactly predictable. For this it is necessary that $\delta x_1 = \delta x_2 \rightarrow 0$. In this case $\delta P(x_1)$ vanishes and no possiblity exists to measure the position of the particle 1. The situation is then the same as in the case of the original EPR wavefunction (5). It is then clear that the product of the uncertainties in P and Q can be much smaller than that required by the Heisenberg Uncertainty Principle. However in order to thus determine Q, use is made of of the precise knowledge of the parameter x_0 of the wavefunction, i.e. exact knowledge of how the wavefunction is prepared is required. But if *a priori* knowledge about wavefunction preparation is admitted, it is trivial to show that observables with non-commuting operators can be simultaneously 'known' with a joint precision far exceeding that allowed by the Heisenberg Uncertainty Relation. To give a concrete example of this, the process of para-positronium annihilation at rest: $e^+e^- \rightarrow \gamma\gamma$ may be considered. The uncertainty in the momentum Δp of one of the decay photons is determined by the mean lifetime of the decay process $\tau = 1.25 \times 10^{-10}$ sec:

$$\Delta p = \frac{h}{c\tau}.$$

The Heisenberg Uncertainty Relation then predicts

$$\Delta x > 3.75cm.$$

The technically simple measurement of the position of the photon in the direction parallel to its momentum to within 1mm (for example, by observing a recoil electron from Compton Scattering of the photon [9]) then allows simultaneous knowledge of the position and momentum of the electron (whose quantum mechanical operators do not commute) with an accuracy $\simeq 40$ times better than 'allowed' by the Heisenberg Uncertainty Relation. Of course the Uncertainty Relation does indeed limit the precision of any attempt to *simultaneously measure* a pair of non-commuting observables. However, as the counter example given above shows, it does not apply to *a priori* knowledge from state preparation, as used by EPR in the discussion of their gedanken experiment. There is therefore nothing remarkable (certainly no 'paradox') in the fact that non-commuting observables can be 'known' more accurately than allowed by the Uncertainty Relation if information about state preparation is also included, as is the case for the EPR gedanken experiment.

It has been stressed above, that no meaningful conclusions can be drawn from any gedanken experiment based upon non square-integrable wave functions. A similar criticism was made by Johansen [10] concerning a paper of Bell [11] where the erroneous conclusion was drawn, by the use of a non square-integrable wave function, that states with a positive Wigner distribution (as is in fact the case for the EPR wave function (5)) necessarily yield a local hidden variable model. A corollary is given by the 'complementary' limits discussed by Bohr [2], where an aspect of classical physics is recovered, yielding a precise position or momentum for a particle. Such exact limits are of limited physical interest since the corresponding wavefuctions are not square integrable for the conjugate variable, and so can have no physical interpretation within quantum mechanics. The Dirac δ-function is a calculational device of extreme utility. It should never be forgotten, however, that it is only a mathematical idealisation never realised in the wavefunction of any actual physical system.

Acknowledgements

I thank N.Gisin and D.J.Moore for reading this paper and for their critical comments.

References

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[5] Almost all subsequent discussion of the 'EPR experiments' in the literature is, instead, based on Bohm's gedanken experiment involving correlated spin measurements. See D.Bohm, 'Quantum Theory' Prentice Hall Inc, New York 1952, Chapter XXII.

[6] i.e. relacing in the statement of EPR the hypothesis QMTC by QMT.

[7] Each hypothesis is assumed to be either true or false. The symbols $\otimes$ and $\oplus$ denote, respectively logical 'and' and 'or'. A bar on a logical variable indicates negation.

[8] i.e. that x_1 lies between $x_1 - \delta x_1/2$ and $x_1 + \delta x_1/2$

[9] See, for example L.R.Kasday, J.D.Ullman and C.S.Wu, Il Nuovo Cimento **25 B** (1975) 633.

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